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Brief paper

Output feedback variable structure control for linear systems with uncertainties and disturbances ☆

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Abstract

This paper proposes a dynamic output feedback variable structure controller for linear MIMO systems with mismatched and matched norm-bounded uncertainties and matched nonlinear disturbances. The proposed controller consists of nonlinear and linear parts, similar to the *unit vector* type of controller. The nonlinear part of the new controller takes care only of *matched* uncertainties and disturbances and, on the other hand, the linear part with *full dynamics* completely handles *mismatched* uncertainties. Designing such a linear part leads to use a Lyapunov function associated with the full states, which achieves the *global* stability against the *mismatched* uncertainties. The resulting criteria are, furthermore, converted into solvable ones, by using the so-called cone complementary linearization algorithm for bi-convex problems.

Keywords: Variable structure control; Dynamic output feedback; Mismatched uncertain systems; Linear matrix inequalities.

1. Introduction

Since the 1970s, variable structure control (VSC) has been ranked as one of the most popular controls among control engineers (Decarlo, Zak, & Matthews, 1988; Hung, Gao, & Hung, 1993; Utkin, 1997), who have applied the VSC theory for various practical systems with uncertainties, input uncertainties, exogenous disturbances, or mixed uncertainties at the same time.

In the literature of output feedback VSC which consists of nonlinear and linear parts for systems with disturbances and/or uncertainties, there have been fundamentally two approaches to design the linear part under the "output feedback" feature. The first one is to use state-observers (Emelyanov, Korovin, Nersisian, & Nisenzon, 1992; Zak & Hui, 1993) and the second one, direct output-based controllers such as static gains types (Choi, 2002; Hsu & Lizarralde, 1998; Kwan, 1996, 2001; Zak & Hui, 1993) and dynamic compensators types (Bag, Spurgeon, & Edwards, 1997; Edwards & Spurgeon, 1998; Edwards, Spurgeon, & Hebden, 2003; El-Khazali & Decarlo, 1992). Emelyanov et al. (1992) have proposed an observer to use the very same method as the state feedback VSC. Zak and Hui (1993) have also constructed an observer-based output feedback controller and even a simpler controller with a static output feedback structure. Kwan (1996, 2001) and Hsu and Lizarralde (1998) have maintained the linear part as simple as possible, and instead introduced dynamics into the nonlinear part, which allowed them to handle a larger class of matched uncertainties. El-Khazali and Decarlo (1992), Bag et al. (1997), Edwards and Spurgeon (1998) and Edwards et al. (2003) have considered dynamic variable structure compensators. Especially, Edwards and Spurgeon (1998) and Edwards et al. (2003) have systematically developed a switching surface design method using a dynamic compensator. However, all these methods are not applicable to systems with *mismatched* uncertainties. When they present the linear part, they simply focused on achieving zero-attraction on the switching surface under the assumption that the system already converged to

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the switching surface. But, if there are *mismatched* uncertainties in the system, one cannot *a priori* ensure that the system converges to the switching surface.

More recently, a conceptual innovation for mismatched uncertainties has been proposed by Choi (2002) with using the notion of ultimate boundedness of Corless and Leitmann (1981) and a 'unit vector' type of controller such as Ryan and Corless (1984). In this work, Choi (2002) chose a Lyapunov function related to all of the states to ensure global stability. In particular, he constructed the linear part with a static gain robust to mismatched uncertainties and the nonlinear part with a certain *dynamics* to handle matched uncertainties, disturbances, and some terms arising from the simple structure of the "static" output feedback gain.

In this paper, we shall follow the innovation for mismatched uncertainties by Choi (2002). The difference is that we shall focus on the linear part rather than the nonlinear part. For the linear part, we construct full dynamics to handle the "output feedback" feature and *mismatched* uncertainties at the same time, while to handle *matched* uncertainties and disturbances, we maintain the nonlinear part as simple as possible. This kind of controller is henceforth called as dynamic output feedback variable structure controller (DOF-VSC). In addition, the structural proposition of the controller, which presents much more flexibility on the linear part than on the nonlinear one, allows us to consider its applications of MIMO systems. The potential advantages include the use of various robust control techniques when designing the linear part of controller to deal with performance-oriented, saturation-related, or their mixed problems. The resulting stability criteria are indeed non-convex because we want to handle the mismatched and matched norm-bounded uncertainties simultaneously. Using the so-called "cone complementary linearization" algorithm (El Ghaoui, Oustry, & AitRami, 1997), which is an efficient algorithm to solve bi-convex problems, we shall propose an LMI-like formulation for finding feasible solutions. The LMI-like formulation is an iterative three-step procedure, in which each step consists of a sufficient convex condition.

Section 2 describes the target system and the new DOF-VSC. Then, Section 3 suggests main stabilizing criteria. Section 4 presents design algorithms for the criteria in terms of LMI-like conditions. By two examples, Section 5 demonstrates the performance of the proposed DOF-VSC using the semi-definite programming. In symmetric block matrices, we use $\|x\|$ as the 2-norm of vector x or $\|X\|$ as the induced 2-norm of matrix X. The symbol $\oplus$ denotes the block diagonal operator.

2. Problem statement

Consider the following mismatched uncertain system:

$$\dot{x}(t) = \{A + \Delta A(t)\}x(t) + \{B + \Delta B(t)\}u(t) + f(t),$$
$$y(t) = Cx(t), \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $y(t) \in R^p$ is the output, $\Delta A(t)$, $\Delta B(t)$, and $f(t)$ represent the system characteristic matrix uncertainty, the input matrix uncertainty, and the nonlinear disturbance, respectively. For the uncertainties and nonlinear disturbance, we assume that the following assumptions are valid:

- (A1): $\Delta A(t)$ is unknown but mismatched norm-bounded time-varying uncertainty

$$\Delta A(t) = H\Delta_1(t)E_1, \quad \|\Delta_1(t)\| \leqslant \gamma_1, \quad t \geqslant 0, \tag{2}$$

  where H, and E_1 are known constant real matrices with appropriate dimensions, and $\Delta_1(t)$ is an unknown matrix function.
- (A2): $\Delta B(t)$ is unknown but matched norm-bounded time-varying uncertainty

$$\Delta B(t) = B\Delta_2(t)E_2, \quad t \geqslant 0,$$
$$\|\Delta_2(t)\| \leqslant \gamma_2, \quad \|\Delta_2(t)E_2\| \leqslant \psi < 1, \tag{3}$$

  where E_2 is a known constant real matrix with an appropriate dimension, $\Delta_2(t)$ is an unknown matrix function, and ψ is a known non-negative constant.
- (A3): The nonlinear disturbance $f(t)$ satisfies the so-called matching condition, i.e.,

$$f(t) = Bw(x,t), \quad \|w(x,t)\| \leqslant b\|x\| + \varepsilon(y,t), \tag{4}$$

  where $b \geqslant 0$ is a known constant and $\varepsilon(y,t) \geqslant 0$ is a known scalar-valued function.
- (A4): $rank(CB) = m$ and the invariant zeros of (A, B, C) $\subset \mathscr{C}^-$, where $\mathscr{C}^-$ denotes the left half space of the complex plane.

For the mismatched uncertain system (1), we design the following DOF-VSC, which is the goal of this paper:

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t),$$
$$u_c(t) = C_c x_c(t) + D_c y(t),$$
$$u(t) = u_c(t) + \bar{u}(x_c, y, t), \tag{5}$$

where $x_c(t) \in R^n$ is the controller state, A_c, B_c, C_c, and D_c are constant controller gain matrices with appropriate dimensions. And $u_c(t)$ denotes the linear part of controller and $\bar{u}(x_c, y, t) \in R^m$ the nonlinear part which is related to the switching control component.

3. Stabilizing criteria

Based on the characteristics of norm-bounded uncertainties, the mismatched uncertain system (1) can be represented via the auxiliary input–output connections:

$$\dot{x}(t) = Ax(t) + Bu_c(t) + B(I + \Delta_2(t)E_2)\bar{u}(x_c, y, t)$$
$$\quad + Bw(x,t) + Hp_1(t) + Bp_2(t),$$
$$q_1(t) = E_1 x(t), \quad q_2(t) = E_2 u_c(t), \quad y(t) = Cx(t) \tag{6}$$

with $p_1(t)=\Delta_1(t)q_1(t)$ and $p_2(t)=\Delta_2(t)q_2(t)$. Consequently, the resulting closed-loop system has the form

$$\dot{\xi}(t)=A_{\rm cl}(\Sigma)\xi(t)+B_{\rm cl}\{\{I+\Delta_2(t)E_2\}\bar{u}(x_{\rm c},y,t)+w(x,t)\}+H_{\rm cl}p(t),$$
$$q(t)=E_{\rm cl}(\Sigma)\xi(t),\quad p(t)=\Delta(t)q(t), \tag{7}$$

where $\Delta(t)=\mathrm{diag}\{\Delta_1(t),\Delta_2(t)\}$, $A_{\rm cl}(\Sigma)=\bar{A}+\bar{B}\Sigma\bar{C}$, $E_{\rm cl}(\Sigma)=\bar{E}_1+\bar{E}_2\Sigma\bar{C}$, and

$$\xi(t)=\begin{bmatrix}x(t)\\x_{\rm c}(t)\end{bmatrix},\quad p(t)=\begin{bmatrix}p_1(t)\\p_2(t)\end{bmatrix},\quad q(t)=\begin{bmatrix}q_1(t)\\q_2(t)\end{bmatrix},$$
$$\Sigma=\begin{bmatrix}D_{\rm c}&C_{\rm c}\\B_{\rm c}&A_{\rm c}\end{bmatrix},\quad B_{\rm cl}=\begin{bmatrix}B\\0\end{bmatrix},\quad H_{\rm cl}=\begin{bmatrix}H&B\\0&0\end{bmatrix},$$
$$\bar{A}=\begin{bmatrix}A&0\\0&0\end{bmatrix},\quad \bar{B}=\begin{bmatrix}B&0\\0&I\end{bmatrix},\quad \bar{C}=\begin{bmatrix}C&0\\0&I\end{bmatrix},$$
$$\bar{E}_1=\begin{bmatrix}E_1&0\\0&0\end{bmatrix},\quad \bar{E}_2=\begin{bmatrix}0&0\\E_2&0\end{bmatrix}.$$

We shall now consider quadratic Lyapunov stability with a candidate Lyapunov function $V(\xi(t))$, mapping from R^{2n} to R

$$V(\xi(t))\triangleq\xi^{\rm T}(t)P\xi(t),\quad P=P^{\rm T}=\begin{bmatrix}X&Y\\Y^{\rm T}&Z\end{bmatrix}\in R^{2n}, \tag{8}$$

where P is a positive constant matrix. To guarantee the Lyapunov stability, the derivative of this function should satisfy the following inequality condition:

$$\dot{V}(\xi)=2\xi^{\rm T}P\dot{\xi}(t)<0 \tag{9}$$

along with the norm-bounded constraint for all p, q, and $\lambda_1>0$, $\lambda_2>0$

$$p^{\rm T}(\lambda_1 I\oplus\lambda_2 I)p\leqslant q^{\rm T}(\lambda_1\gamma_1^2 I\oplus\lambda_2\gamma_2^2 I)q,$$
$$\|\Delta_i\|\leqslant\gamma_i,\quad i=1,2,\ \|\Delta_2E_2\|\leqslant\psi<1. \tag{10}$$

Here, the nonlinear part of the input, $\bar{u}$, will be chosen to handle the terms associated with the matched uncertainties and disturbances, $\xi^{\rm T}P\{B_{\rm cl}\{I+\Delta_2E_2\}\bar{u}+B_{\rm cl}w\}$, which makes us choose a new switching function as follows:

$$\sigma(x_{\rm c},y,t)\triangleq B_{\rm cl}^{\rm T}P\xi(t)\equiv B^{\rm T}Xx(t)+B^{\rm T}Yx_{\rm c}(t). \tag{11}$$

However, the "output feedback" feature does not allow us to use $x(t)$ to construct the switching function. Therefore, we shall impose the following equality (hard) condition:

$$B^{\rm T}X\equiv GC, \tag{12}$$

where G is a constant matrix with an appropriate dimension. In this case, the switching function becomes

$$\sigma(x_{\rm c},y,t)\triangleq Gy(t)+B^{\rm T}Yx_{\rm c}(t), \tag{13}$$

In the following theorem, based on the new DOF-VSC (5), we shall present a new stabilizable criterion for the closed-loop system (7).

Theorem 1. *Based on all assumptions, let us choose the nonlinear part of the controller via a switching function* $\sigma(x_{\rm c},y,t)\triangleq Gy(t)+B^{\rm T}Yx_{\rm c}(t)$ *as*

$$\bar{u}(x_{\rm c},y,t)=-\frac{b^2}{2(1-\psi)\zeta}\sigma(x_{\rm c},y,t)-\frac{\varepsilon(y,t)}{(1-\psi)\|\sigma(x_{\rm c},y,t)\|}\sigma(x_{\rm c},y,t), \tag{14}$$

where ζ *and* G *are values in the following condition. If there exist matrices*

$$P\triangleq\begin{bmatrix}X&Y\\Y^{\rm T}&Z\end{bmatrix}>0,\quad \Sigma,G,\ \lambda_1>0,\ \lambda_2>0,\ \zeta>0 \tag{15}$$

satisfying the following constrained inequalities:

$$0>M(\Sigma,P,\lambda_1,\lambda_2,\zeta)\triangleq\begin{bmatrix}(*)&PH_{\rm cl}\\H_{\rm cl}^{\rm T}P&-(\lambda_1I\oplus\lambda_2I)\end{bmatrix}, \tag{16}$$

$$GC=B^{\rm T}X, \tag{17}$$

$$(*)\triangleq A_{\rm cl}^{\rm T}(\Sigma)P+PA_{\rm cl}(\Sigma)+E_{\rm cl}^{\rm T}(\Sigma)(\lambda_1\gamma_1^2I\oplus\lambda_2\gamma_2^2I)E_{\rm cl}(\Sigma)+(\zeta I\oplus 0),$$

then the closed-loop system (7) *is globally asymptotically stable.*

Proof. By the S-procedure (Boyd, El Ghauoi, Feron, & Balakrishnan, 1994), combining conditions (9) and (10) provides

$$\begin{aligned}\dot{V}(\xi)&\leqslant 2\xi^{\rm T}P\{A_{\rm cl}\xi+B_{\rm cl}\{(I+\Delta_2E_2)\bar{u}+w\}+H_{\rm cl}p\}\\&\quad-p^{\rm T}(\lambda_1I\oplus\lambda_2I)p+q^{\rm T}(\lambda_1\gamma_1^2I\oplus\lambda_2\gamma_2^2I)q\\&=2\xi^{\rm T}P\{A_{\rm cl}\xi+H_{\rm cl}p\}-p^{\rm T}(\lambda_1I\oplus\lambda_2I)p\\&\quad+\xi^{\rm T}E_{\rm cl}^{\rm T}(\Sigma)(\lambda_1\gamma_1^2I\oplus\lambda_2\gamma_2^2I)E_{\rm cl}^{\rm T}(\Sigma)\xi\\&\quad+2\sigma^{\rm T}(x_{\rm c},y,t)\{(I+\Delta_2E_2)\bar{u}+w\}.\end{aligned} \tag{18}$$

Using (A2) and (A3), and (14), we can get that

$$\begin{aligned}&2\sigma^{\rm T}(x_{\rm c},y,t)\{(I+\Delta_2E_2)\bar{u}+w\}\\&\leqslant 2\sigma^{\rm T}(x_{\rm c},y,t)\bar{u}+2\|\sigma(x_{\rm c},y,t)\|\|\Delta_2E_2\|\|\bar{u}\|\\&\quad+2\|\sigma(x_{\rm c},y,t)\|\{b\|x\|+\varepsilon\}\\&=-\frac{b^2}{(1-\psi)\zeta}\|\sigma(x_{\rm c},y,t)\|^2-\frac{2\varepsilon}{(1-\psi)}\|\sigma(x_{\rm c},y,t)\|\\&\quad+\frac{b^2\psi}{(1-\psi)\zeta}\|\sigma(x_{\rm c},y,t)\|^2+\frac{2\varepsilon\psi}{(1-\psi)}\|\sigma(x_{\rm c},y,t)\|\\&\quad+2\|\sigma(x_{\rm c},y,t)\|\{b\|x\|+\varepsilon\}\\&=-\frac{b^2}{\zeta}\|\sigma(x_{\rm c},y,t)\|^2+2b\|\sigma(x_{\rm c},y,t)\|\|x\|\\&\leqslant\zeta\|x\|^2.\end{aligned}$$

Therefore, we have the resulting condition such that

$$\dot{V}(\xi) \leqslant 2\xi^{\mathrm{T}} P\{A_{\mathrm{cl}}\xi + H_{\mathrm{cl}}p\} - p^{\mathrm{T}}(\lambda_1 I \oplus \lambda_2 I)p + \xi^{\mathrm{T}} E_{\mathrm{cl}}^{\mathrm{T}}(\Sigma)(\lambda_1\gamma_1^2 I \oplus \lambda_2\gamma_2^2 I)E_{\mathrm{cl}}^{\mathrm{T}}(\Sigma)\xi + \zeta\|x\|^2 = \begin{bmatrix} \xi \\ p \end{bmatrix}^{\mathrm{T}} M(\Sigma, P, \lambda_1, \lambda_2, \zeta) \begin{bmatrix} \xi \\ p \end{bmatrix}.$$

Consequently, based on equality (17), if the inequality (16) is verified, the closed-loop system (7) is globally asymptotically stable. □

Remark. As mentioned in Choi (2002), the switching component (13) is derived to help showing the Lyapunov stability condition $\dot{V}(\xi) < 0$ instead of the usual reachability condition $\sigma^{\mathrm{T}}\dot{\sigma} < 0$. Thus, with the proposed DOF-VSC (5) and (13), we can guarantee the global asymptotic stability of the closed-loop system (7) but cannot assure that σ tends to zero in finite time.

A special case of system (1) where the nonlinear disturbance is a function of the only output $y(t)$ and is bounded by a known function $\varepsilon(y, t)$ of the output, we propose another DOF-VSC with the following stabilizing criterion:

- (A5): The nonlinear disturbance $f(t)$ satisfies the matching conditions, i.e.,

$$f(t) = Bw(y, t), \quad \|w(y, t)\| \leqslant \varepsilon(y, t), \tag{19}$$

  where $\varepsilon(y, t) \geqslant 0$ is a known scalar-valued function.

Theorem 2. *Based on* (A1), (A2), (A4), *and* (A5), *let us choose the nonlinear part of controller via a switching function* $\sigma(x_c, y, t) \triangleq Gy(t) + B^{\mathrm{T}}Yx_c(t)$ *as*

$$\bar{u}(x_c, y, t) = -\frac{\varepsilon(y, t)}{(1-\psi)\|\sigma(x_c, y, t)\|}\sigma(x_c, y, t). \tag{20}$$

If there exist matrices

$$P \triangleq \begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix} > 0, \quad \Sigma, \quad G, \quad \lambda_1 > 0, \quad \lambda_2 > 0 \tag{21}$$

satisfying the following constrained inequalities:

$$\begin{bmatrix} A_{\mathrm{cl}}^{\mathrm{T}}(\Sigma)P + PA_{\mathrm{cl}}(\Sigma) + E_{\mathrm{cl}}^{\mathrm{T}}(\Sigma)(\lambda_1\gamma_1^2 I \oplus \lambda_2\gamma_2^2 I)E_{\mathrm{cl}}(\Sigma) & PH_{\mathrm{cl}} \\ H_{\mathrm{cl}}^{\mathrm{T}}P & -(\lambda_1 I \oplus \lambda_2 I) \end{bmatrix} < 0, \tag{22}$$

$$B^{\mathrm{T}}X = GC, \tag{23}$$

the closed-loop system (7) *is globally asymptotically stable.*

Proof. We omit the proof, which is straightforward from that of Theorem 1. □

In this section, we proposed two new stabilizing criteria for the mismatched uncertain systems with the proposed new DOF-VSC. Since the resulting problems are not convex, we shall propose solvable formulations in the following section.

4. Controller design using LMI-like conditions

We can remove one variable, λ_1, in (16) and (17) by scaling all other variables with λ_1

$$\lambda_1^{-1}P \Rightarrow P, \quad \lambda_1^{-1}\lambda_1 \Rightarrow 1, \quad \lambda_1^{-1}\lambda_2 \Rightarrow \lambda_2, \quad \lambda_1^{-1}\zeta \Rightarrow \zeta, \quad \lambda_1^{-1}G \Rightarrow G. \tag{24}$$

There are, however, still two difficulties in conditions (16) and (17) in Theorem 1. First, equality (17) contains a hard constraint. This condition, in fact, can be handled with a simple relaxation technique into the following LMI condition:

$$0 < \mathscr{M}_1(X, G, \gamma) \triangleq \begin{bmatrix} \gamma I & B^{\mathrm{T}}X - GC \\ XB - C^{\mathrm{T}}G^{\mathrm{T}} & \gamma I \end{bmatrix}, \tag{25}$$

where γ will be sent to an extremely small number related to the computational precision unit. Second, inequality (16) of Theorem 1 is non-convex because there are product terms involving the free variable P and the controller parameter Σ. For solving the non-convex problem of Theorem 1, therefore, we shall simplify the condition (16) via the elimination lemma (Boyd et al., 1994) for the free variable Σ. The resulting conditions can be written as, with some efforts involving the Schur complement equivalence

$$0 > \mathscr{M}_2(X, \lambda_2, \zeta^{-1}), \tag{26}$$

$$0 > \mathscr{M}_3(\bar{X}, \lambda_2^{-1}, \zeta^{-1}), \tag{27}$$

where $\mathscr{M}_2(X, \lambda_2, \zeta^{-1})$ and $\mathscr{M}_3(\bar{X}, \lambda_2^{-1}, \zeta^{-1})$ denote

$$\mathscr{M}_2(\cdot) \triangleq \begin{bmatrix} (1,1)_a & (C_\perp^{\mathrm{T}})^{\mathrm{T}}XH & (C_\perp^{\mathrm{T}})^{\mathrm{T}}XB & (C_\perp^{\mathrm{T}})^{\mathrm{T}}E_1^{\mathrm{T}} & (C_\perp^{\mathrm{T}})^{\mathrm{T}} \\ H^{\mathrm{T}}XC_\perp^{\mathrm{T}} & -I & 0 & 0 & 0 \\ B^{\mathrm{T}}XC_\perp^{\mathrm{T}} & 0 & -\lambda_2 I & 0 & 0 \\ E_1C_\perp^{\mathrm{T}} & 0 & 0 & -\gamma_1^{-2}I & 0 \\ C_\perp^{\mathrm{T}} & 0 & 0 & 0 & -\zeta^{-1}I \end{bmatrix},$$

$$\mathscr{M}_3(\cdot) \triangleq \begin{bmatrix} (1,1)_b & N_1^{\mathrm{T}}\bar{X}E_1^{\mathrm{T}} & N_1^{\mathrm{T}}\bar{X} \\ E_1\bar{X}N_1 & -\gamma_1^{-2}I & 0 \\ \bar{X}N_1 & 0 & -\zeta^{-1}I \end{bmatrix},$$

$$(1,1)_a \triangleq (C_\perp^{\mathrm{T}})^{\mathrm{T}}(A^{\mathrm{T}}X + XA)C_\perp^{\mathrm{T}},$$

$$(1,1)_b \triangleq N_1^{\mathrm{T}}(\bar{X}A^{\mathrm{T}} + A\bar{X} + HH^{\mathrm{T}} + \lambda_2^{-1}BB^{\mathrm{T}})N_1 - \lambda_2^{-1}\gamma_2^{-2}N_2^{\mathrm{T}}N_2.$$

We now remark two points. The first is the well-known completion lemma (Packard, Zhou, Pandey, & Becker, 1991) where the existence condition of a positive definite matrix P can be written with X and $\bar{X}$ like

$$0 \leqslant \mathscr{M}_4(X, \bar{X}) \triangleq \begin{bmatrix} X & I \\ I & \bar{X} \end{bmatrix}, \tag{28}$$

which is in fact equal to the condition that $X \geqslant \bar{X}^{-1} > 0$ and thus one can construct Y and Z from the relation that $X - \bar{X}^{-1} = YZ^{-1}Y^{\mathrm{T}}$. The second is that these conditions (26) and (27) are non-convex, which causes some difficulty in finding a feasible solution set of $X, \bar{X}, \lambda_2, \zeta^{-1}$. However, these kind of problems, which belong to bi-convex, or especially, bilinear matrix inequality (BMI) conditions, are well-studied and several algorithms have been proposed in the literature. Since we need to handle only one scalar variable λ_2 in (26) and (27) that appears nonlinearly in both the conditions, the best algorithm may be the cone complementary linearization algorithm (El Ghaoui et al., 1997). Applying the cone complementary linearization algorithm for this case, we shall define $\bar{\lambda}_2$ as λ_2^{-1} in (27) and additionally construct one more condition

$$0 \leqslant \mathscr{M}_5(\lambda_2, \bar{\lambda}_2) \triangleq \begin{bmatrix} \lambda_2 & 1 \\ 1 & \bar{\lambda}_2 \end{bmatrix}. \tag{29}$$

The final goal is to find feasible λ_2 and $\bar{\lambda}_2$ yielding $\lambda_2\bar{\lambda}_2 = 1$. But the feasible solutions λ_2 and $\bar{\lambda}_2$ yielding all the LMI conditions (26), (27), (28), and (29) simply provides the relation that $\lambda_2\bar{\lambda}_2 \geqslant 1$. Therefore, the cone complementary linearization algorithm (El Ghaoui et al., 1997) runs the alternative procedure, to reach the equality condition that $\lambda_2\bar{\lambda}_2 = 1$, that minimizes λ_2 or $\bar{\lambda}_2$ with respect to a fixed λ_2 or to a fixed $\bar{\lambda}_2$, respectively, and alternatively. This algorithm performs like an almost convex algorithm, that is, we called an LMI-like formulation, when only one variable is involved.

Based on this cone complementary linearization algorithm, therefore, we propose the following algorithm to solve the conditions in Theorem 1.

Proposition. *A feasible solution of the conditions* (16) *in Theorem* 1 *can be found via the following three-step procedure*.

- *Step* 1 (initialization): *Find a feasible set of* $X, G, \gamma, \lambda_2, \zeta^{-1}, \bar{X}$, *and* $\bar{\lambda}_2$ *yielding*

  $$0 < \mathscr{M}_1(X, G, \gamma), \quad 0 > \mathscr{M}_2(X, \lambda_2, \zeta^{-1}),$$
  $$0 > \mathscr{M}_3(\bar{X}, \bar{\lambda}_2, \zeta^{-1}), \quad 0 \leqslant \mathscr{M}_4(X, \bar{X}),$$
  $$0 \leqslant \mathscr{M}_5(\lambda_2, \bar{\lambda}_2),$$

  where $\mathscr{M}_i$ *are defined in* (25), (26), (27), (28) *and* (29).
- *Step* 2 (iteration to reach σ-feasibility): *For a fixed* $\bar{\lambda}_2$ *as the value obtained in the previous step*, *minimize* $\bar{\sigma} \triangleq (\bar{\lambda}_2\lambda_2 + \gamma)$ *subject to the above conditions with respect to all other variables* $X, G, \gamma, \lambda_2, \zeta^{-1}$, *and* $\bar{X}$. *If* $\bar{\sigma}$ *is less than* $(1+\sigma)$, *go to the next step. Otherwise*, *fix* λ_2 *with the value found in this step. Then minimize* $\bar{\sigma} \triangleq (\bar{\lambda}_2\lambda_2 + \gamma)$ *subject to the above conditions with respect to all other variables* X, G, γ, $\bar{\lambda}_2$, ζ^{-1}, *and* $\bar{X}$. *If* $\bar{\sigma}$ *is less than* $(1+\sigma)$, *go to the next step. Otherwise*, *repeat Step* 2. *Here*, *the choice of the decision criterion* σ *depends on the numerical precision of the computer*, *which is extremely small.*
- *Step* 3 (Σ construction): *For* σ*-feasible* X *and* $\bar{X}$, *perform the singular decomposition of the matrix* $(X - \bar{X}^{-1})$ *to obtain* Y *and* Z *from the relation that* $(X - \bar{X}^{-1} = YZ^{-1}Y^{\mathrm{T}})$ *and thus to form the* P *in* (16). *With respect to all the values obtained in the previous step*, *then*, *the inequality in* (16) *is linear in* Σ. *Therefore*, *by using semi-definite programming*, *one can get the linear part of the controller*, Σ.

We use the terminology of "σ-feasibility," rather than "feasibility," because numerical algorithms usually work in the computational precision. We remark that the algorithm for Theorem 2 can be obtained if one replaces $\mathscr{M}_2(X, \lambda_2, \zeta^{-1})$ and $\mathscr{M}_3(\bar{X}, \bar{\lambda}_2, \zeta^{-1})$ in the above proposition with

$$\bar{\mathscr{M}}_2(X, \lambda_2) \triangleq \begin{bmatrix} (1,1)_{\mathrm{a}} & (C_\perp^{\mathrm{T}})^{\mathrm{T}}XH & (C_\perp^{\mathrm{T}})^{\mathrm{T}}XB & (C_\perp^{\mathrm{T}})^{\mathrm{T}}E_1^{\mathrm{T}} \\ H^{\mathrm{T}}XC_\perp^{\mathrm{T}} & -I & 0 & 0 \\ B^{\mathrm{T}}XC_\perp^{\mathrm{T}} & 0 & -\lambda_2 I & 0 \\ E_1C_\perp^{\mathrm{T}} & 0 & 0 & -\gamma_1^{-2}I \end{bmatrix},$$

$$\bar{\mathscr{M}}_3(\bar{X}, \bar{\lambda}_2) \triangleq \begin{bmatrix} (1,1)_{\mathrm{b}} & N_1^{\mathrm{T}}\bar{X}E_1^{\mathrm{T}} \\ E_1\bar{X}N_1 & -\gamma_1^{-2}I \end{bmatrix},$$

$$(1,1)_{\mathrm{a}} \triangleq (C_\perp^{\mathrm{T}})^{\mathrm{T}}(A^{\mathrm{T}}X + XA)C_\perp^{\mathrm{T}},$$

$$(1,1)_{\mathrm{b}} \triangleq N_1^{\mathrm{T}}(\bar{X}A^{\mathrm{T}} + A\bar{X} + HH^{\mathrm{T}} + \bar{\lambda}_2BB^{\mathrm{T}})N_1 - \bar{\lambda}_2\gamma_2^{-2}N_2^{\mathrm{T}}N_2.$$

5. Numerical example

In this section, we shall demonstrate the performance of the proposed DOF-VSC with an example based on the systems, suggested by Choi (2002).

Example (*Choi, 2002*). Consider a mismatched uncertain system (1) including mismatched norm-bounded time-varying uncertainties with the following data:

$$A = \begin{bmatrix} -3 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\Delta A(t) = \begin{bmatrix} r_1(t) & 0 & r_2(t) \\ 0 & 0 & 0 \\ 0 & r_4(t) & 0 \end{bmatrix}, \quad \Delta B(t) = 0.5Br_5(t),$$

$$f(y,t) = B(r_3(t)[0\ 1]y(t) + r_6(t)), \quad |r_i(t)| \leqslant 1. \tag{30}$$

We can get the additional information for the proposed new DOF-VSC from the given data (30) as follows:

$$H = E_1 = I, \quad \Delta_1(t) = \Delta A(t), \quad E_2 = 0.5, \quad \Delta_2(t) = r_5(t),$$

$$\psi = 0.5, \quad \gamma_1 = \gamma_2 = 1, \quad \varepsilon(y,t) \triangleq 1 + |y_2(t)|.$$

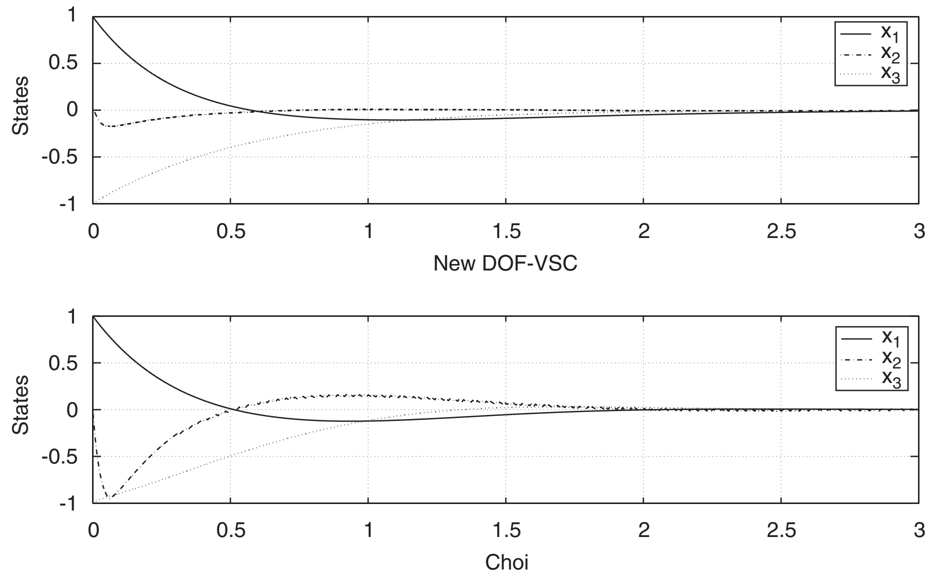

Fig. 1. State responses.

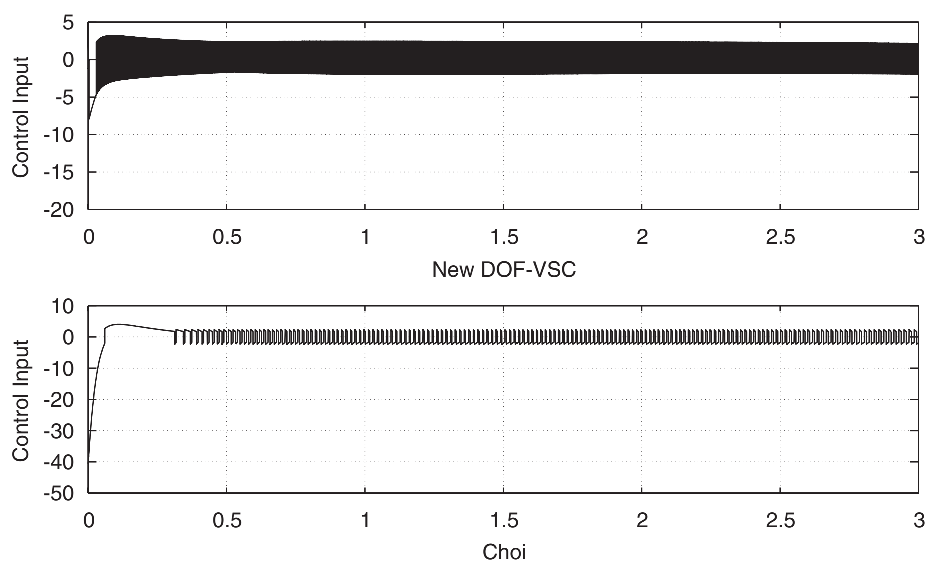

Fig. 2. Control inputs.

In this case, since the nonlinear disturbance of this example was bounded by a known function of the output, $1 + |y_2(t)|$, we calculated the DOF-VSC with (20) based on Theorem 2. And then, we compared the simulation results with the results of Choi (2002) for the initial condition $x(0) = [10-1]^{\mathrm{T}}$, where $r_i(t) = \sin(t)$. The states and the control inputs are shown by Figs. 1 and 2. Fig. 2 shows that the proposed control makes the system reach the switching surface faster than Choi's control. We guess that the full dynamics in the linear part of the controller may have much more potential to carry out good transient behaviors than the static gain.

Now let us show the performance of the *full-dynamic* output feedback feature, by calculating the allowable maximum robust bounds of γ_1 related to mismatched uncertainties of ours. By the LMI toolbox in the Matlab 6.5, we calculated the maximized robust bounds of γ_1 of Theorem 2 for (30) in Table 1.

Table 1
Results of the performance

Maximum robust bounds of "γ_1"	
New DOF-VSC	2.073267

6. Concluding remarks

For output feedback systems, we proposed a new DOF-VSC which consists of a linear part with full dynamics and a nonlinear part with an additional VSC component. The VSC part of the proposed DOF-VSC was designed with the simplest form to handle only matched uncertainties and disturbances. The full dynamic output feedback control part was designed to achieve the global asymptotic stability of the closed-loop system against not only the mismatched uncertainties but also a certain term *a posteriori* arising from simplification of the nonlinear control. We remark that the choice of the Lyapunov function associated with the full states, rather than with the switching-surface vector, played a crucial role in designing the proposed controller. Then, for the resulting non-convex criteria, we presented an iterative LMI-like formulation. If we apply the proposed control methodology for performance-oriented problems such as guaranteed cost, tracking, saturations, and their combinations, we believe that the proposed control methodology may produce better results than other methodologies in the literature, because the full dynamic structure of the linear part allows for huge room for other problems.

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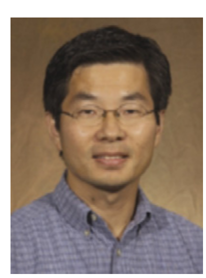

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